

RESIDUAL CAUCHY-TYPE FORMULA ON RIEMANN SURFACES.

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ABSTRACT. We construct a Cauchy type formula on open subdomains of Riemann surfaces of the form $V = \{z \in \mathbb{CP}^2 : P(z) = 0\}$.

1. INTRODUCTION.

Let $V \subset \tilde{V}$ be a smoothly bordered open subset of a Riemann surface

$$\tilde{V} = \{z \in \mathbb{CP}^2 : P(z) = 0\}. \quad (1.1)$$

The goal of the present article is the construction of a Cauchy-type integral formula defining the values of a holomorphic function f on V through its values on the boundary bV . The article represents a continuation of the line of research developed in the joint articles with Gennadi Henkin [HP2] and [HP3] and devoted to application of integral formulas to complex analysis on subvarieties of projective spaces.

Throughout the article we assume that a holomorphic function f , for which we construct a boundary integral formula, is defined on some tubular neighborhood of V in \mathbb{CP}^2 . As in [HP1], we identify f on a domain in \mathbb{CP}^2 with its lift to a domain in $\mathbb{S}^5(1)$ satisfying appropriate homogeneity conditions. Then the sought formula is constructed as the residue of the formula on a tubular domain

$$U^\epsilon = \{z \in \mathbb{S}^5(1) : |P(z)| < \epsilon, \varrho(z) < 0\}, \quad (1.2)$$

where we assume that $V = \{z \in \tilde{V} : \varrho(z) < 0\}$.

Integral formula constructed in the article uses two types of barrier-functions. The first barrier, which is described in the lemma below is a Weil-type barrier constructed by A. Weil in [W] (see also [HP2]).

Lemma 1.1. *Let $P(\zeta)$ be a homogeneous polynomial of variables $\zeta_0, \zeta_1, \zeta_2$ of degree d . Then there exist polynomials $\{Q^i(\zeta, z)\}_{i=0}^2$ satisfying:*

$$\begin{cases} P(\zeta) - P(z) = \sum_{i=0}^2 Q^i(\zeta, z) \cdot (\zeta_i - z_i), \\ Q^i(\lambda\zeta, \lambda z) = \lambda^{d-1} \cdot Q^i(\zeta, z) \text{ for } \lambda \in \mathbb{C}. \end{cases} \quad (1.3)$$

The choice of the second barrier is adjusted to the boundary bV , but also depends on the global structure of the Riemann surface \tilde{V} . The second barrier has the form

$$F(z, \zeta) = \sum_{i=0}^2 R_i(z)(\zeta_i - z_i) \quad (1.4)$$

with coefficients R_i holomorphically depending on z . From this point on we assume that for an arbitrary $z \in V$ there exists a vector field $R(w) = [R_0(w), R_1(w), R_2(w)]$ on V with values in the conormal vector

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bundle of V , satisfying the following conditions

- (i) $R(w) \neq 0$ for any $w \in V$,
- (ii) there exists a neighborhood $\mathcal{V}_z \ni z$ such that for $w \in \mathcal{V}_z$ the set $\mathcal{S}(w) = \{\zeta \in V : F(w, \zeta) = 0\}$ consists of finitely many points $\{w^{(0)} = w, w^{(1)}, \dots, w^{(p)}\}$, at which the line $\{\zeta : F(w, \zeta) = 0\}$ transversally intersects V .

(1.5)

Existence of a vector function $R(w)$ satisfying condition (i) of (1.5) follows from the triviality of the conormal vector bundle over the open Riemann surface V (see [Fo]), and condition (ii) can be achieved without loss of generality with the use of the Bertini's Theorem (see [Ha]) by slightly changing the field R .

Using barriers (1.3) and (1.4) we can now formulate the main theorem of the article.

Theorem 1. *Let $V \subset \{\mathbb{CP}^2 \setminus \{z_0 = 0\}\}$ and U^ϵ be as in (1.1) and (1.2) respectively, and let f be a holomorphic function of negative homogeneity in U^ϵ for some $\epsilon > 0$. Let $z \in V$ be a fixed point, and let $\mathcal{V}_z \ni z$ be a neighborhood of z in V , such that*

- (i) *conditions (1.5) are satisfied,*
- (ii) *function $\frac{w_1}{w_0}$ takes distinct values at the points $\{w, w^{(1)}, \dots, w^{(p)}\}$ of $\mathcal{S}(w)$ for $w \in \mathcal{V}_z$.*

Then for $w \in \mathcal{V}_z$ the following equality holds for the values of f at the points of $\mathcal{S}(w)$:

$$f(w^{(k)}) = \frac{p+1}{p+2} \cdot \frac{\det A_k(w, w^{(1)}, \dots, w^{(p)})}{\det A(w)}, \quad (1.6)$$

where $A(w)$ is the Vandermonde matrix

$$A(w) = \begin{bmatrix} I & I & \dots & I \\ \frac{w_1}{w_0} & \frac{w_1^{(1)}}{w_0} & \dots & \frac{w_1^{(p)}}{w_0} \\ \vdots & \vdots & & \vdots \\ \left(\frac{w_1}{w_0}\right)^p & \left(\frac{w_1^{(1)}}{w_0}\right)^p & \dots & \left(\frac{w_1^{(p)}}{w_0}\right)^p \end{bmatrix}, \quad (1.7)$$

$A_k(w, w^{(1)}, \dots, w^{(p)})$ is the matrix $A(w)$ with the k -th column replaced by the column

$$G_k(w, w^{(1)}, \dots, w^{(p)}) = \frac{2}{(2\pi i)^3} \left(\sum_{j=0}^p \lim_{\epsilon \rightarrow 0} \int_{\Gamma^\epsilon} f(\zeta) \cdot \left(\frac{\zeta_1}{\zeta_0}\right)^k \times \det \left[\frac{Q(\zeta, w^{(j)})}{P(\zeta)} \frac{R(w^{(j)})}{F(w^{(j)}, \zeta)} \frac{\bar{\zeta}}{B(\zeta, w^{(j)})} \right] d\zeta_0 \wedge d\zeta_1 \wedge d\zeta_2 \right), \quad (1.8)$$

and

$$\Gamma^\epsilon = \{z \in \mathbb{S}^5(1) : |P(z)| = \epsilon, \varrho(z) = 0\}. \quad (1.9)$$

Remark 1. Assumption of negativity of the homogeneity of f in Theorem 1 is made just for shortening the resulting formula. If f has homogeneity $\ell \geq 0$, then Theorem 1 can be applied to function $f(z) \cdot z_0^{-\ell-1}$ and the result multiplied by $z_0^{\ell+1}$.

Remark 2. From the formulation of Theorem 1 it can be seen that the Cauchy-type barrier from (1.4) is local with respect to z and global with respect to ζ , which represents an important new ingredient in formula (1.6).

2. INTEGRAL FORMULAS ON DOMAINS IN \mathbb{CP}^2 .

In this section we construct a Cauchy-Weil-Leray type integral formula for functions on a domain U^ϵ in \mathbb{CP}^2 . We start with the Koppelman-type formula from [HP1] (Proposition 1.2), which originally appeared for strictly pseudonvex manifolds in a very important article by Henkin [He2] (Theorem 3.2), and then interpreted in [HP1] for subdomains of such manifolds. Integral formulas of this type have a long history going back to Moisil [Mo], Fueter [Fe], Martinelli [Ma], Bochner [Bo], Koppelman [Kp].

The proposition below is a reformulation of Proposition 1.2 from [HP1].

Proposition 2.1. *Let P be a homogeneous polynomial defining the curve V as in (1.1), and let f be a holomorphic function of homogeneity ℓ on the domain U^ϵ .*

Then the following equality is satisfied for $z \in U^\epsilon$

$$f(z) = L^\epsilon[f](z), \quad (2.1)$$

with

$$L^\epsilon[f](z) = \frac{2}{(2\pi i)^3} \left[\int_{bU^\epsilon \times [0,1]} f(\zeta) \cdot \omega'_0 \left((1-\lambda) \frac{\bar{z}}{B^*(\zeta, z)} + \lambda \frac{\bar{\zeta}}{B(\zeta, z)} \right) \wedge d\zeta \right. \\ \left. + \int_{U^\epsilon} f(\zeta) \cdot \omega'_0 \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \wedge d\zeta \right],$$

where

$$B^*(\zeta, z) = \sum_{j=0}^2 \bar{z}_j \cdot (\zeta_j - z_j), \quad B(\zeta, z) = \sum_{j=0}^2 \bar{\zeta}_j \cdot (\zeta_j - z_j),$$

$$d\zeta = d\zeta_0 \wedge d\zeta_1 \wedge d\zeta_2, \quad \omega'(\eta) = \sum_{k=0}^2 (-1)^k \eta_k \bigwedge_{j \neq k} d\eta_j,$$

and ω'_0 is the $(0,0)$ -component with respect to z of the form ω' .

Remark. The first integral in the formula above is a proper integral taken over the boundary of U^ϵ . The second integral is a singular integral, and, following the interpretation in [He2], should be understood as

$$\lim_{\delta \rightarrow 0} \int_{U^\epsilon} f(\zeta) \cdot \omega'_0 \left(\frac{\bar{\zeta}}{B(\zeta, z^\delta)} \right) \wedge d\zeta,$$

where $z^\delta = z - \delta \cdot \nu(z)$, and $\nu(z)$ is the normal to $\mathbb{S}^5(1)$ at z .

We will transform the right-hand side of equality (2.1) into a Cauchy-Weil-Leray type formula on U^ϵ . For this transformation we need several lemmas.

In the first lemma for convenience in the further computations we eliminate the integration with respect to parameter λ in the right-hand side of (2.1) and transform the formula for operator L^ϵ in Proposition 2.1.

Lemma 2.2. *For operator L^ϵ in Proposition 2.1 the following equality holds:*

$$L^\epsilon[f](z) = \frac{1}{(2\pi i)^3} \left[\int_{bU^\epsilon} f(\zeta) \cdot \det \left[\frac{\bar{z}}{B^*(\zeta, z)} \quad \frac{\bar{\zeta}}{B(\zeta, z)} \quad \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta \right. \\ \left. + \int_{U^\epsilon} f(\zeta) \cdot \det \left[\frac{\bar{\zeta}}{B(\zeta, z)} \quad \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \quad \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta \right]. \quad (2.2)$$

Proof. We use an alternative definition

$$\omega'(\eta) = \sum_{k=0}^2 (-1)^k \eta_k \bigwedge_{j \neq k} d\eta_j = \frac{1}{2} \det \begin{bmatrix} \eta_0 & d\eta_0 & d\eta_0 \\ \eta_1 & d\eta_1 & d\eta_1 \\ \eta_2 & d\eta_2 & d\eta_2 \end{bmatrix}, \quad (2.3)$$

through determinant, which is motivated by the definition used by F. Sommer in [So] in his proof of the Bergman-Weil formula, and then described in the textbook by B. A. Fuks [Fk]. Since then this formula has been used in many articles starting in [He1, Po].

Using expression $\eta(\zeta, z, \lambda) = (1 - \lambda) \frac{\bar{z}}{B^*(\zeta, z)} + \lambda \frac{\bar{\zeta}}{B(\zeta, z)}$ in (2.3) we obtain

$$\begin{aligned} & \det \begin{bmatrix} \eta(\zeta, z, \lambda) & d_{\zeta, \lambda} \eta(\zeta, z, \lambda) & d_{\zeta, \lambda} \eta(\zeta, z, \lambda) \end{bmatrix} \wedge d\zeta \\ &= \left((1 - \lambda) \det \left[\frac{\bar{z}}{B^*(\zeta, z)} \frac{\bar{\zeta} d\lambda}{B(\zeta, z)} \frac{\lambda d\bar{\zeta}}{B(\zeta, z)} \right] + (1 - \lambda) \det \left[\frac{\bar{z}}{B^*(\zeta, z)} \frac{\lambda d\bar{\zeta}}{B(\zeta, z)} \frac{\bar{\zeta} d\lambda}{B(\zeta, z)} \right] \right. \\ & \quad \left. + \lambda \det \left[\frac{\bar{\zeta}}{B(\zeta, z)} \frac{-\bar{z} d\lambda}{B^*(\zeta, z)} \frac{\lambda d\bar{\zeta}}{B(\zeta, z)} \right] + \lambda \det \left[\frac{\bar{\zeta}}{B(\zeta, z)} \frac{\lambda d\bar{\zeta}}{B(\zeta, z)} \frac{-\bar{z} d\lambda}{B^*(\zeta, z)} \right] \right) \wedge d\zeta \\ &= \left(2\lambda(1 - \lambda) d\lambda \wedge \det \left[\frac{\bar{z}}{B^*(\zeta, z)} \frac{\bar{\zeta}}{B(\zeta, z)} \frac{d\bar{\zeta}}{B(\zeta, z)} \right] - 2\lambda^2 d\lambda \wedge \det \left[\frac{\bar{\zeta}}{B(\zeta, z)} \frac{\bar{z}}{B^*(\zeta, z)} \frac{d\bar{\zeta}}{B(\zeta, z)} \right] \right) \wedge d\zeta \\ &= 2\lambda d\lambda \wedge \det \left[\frac{\bar{z}}{B^*(\zeta, z)} \frac{\bar{\zeta}}{B(\zeta, z)} \frac{d\bar{\zeta}}{B(\zeta, z)} \right] \wedge d\zeta, \end{aligned}$$

and

$$\omega'_0(\eta) \wedge d\zeta = \frac{1}{2} \cdot 2\lambda d\lambda \wedge \det \left[\frac{\bar{z}}{B^*(\zeta, z)} \frac{\bar{\zeta}}{B(\zeta, z)} \frac{d\bar{\zeta}}{B(\zeta, z)} \right] \wedge d\zeta.$$

Then we have

$$\begin{aligned} \int_0^1 \omega'_0(\eta(\zeta, z, \lambda)) \wedge d\zeta &= \left(\int_0^1 \lambda d\lambda \right) \det \left[\frac{\bar{z}}{B^*(\zeta, z)} \frac{\bar{\zeta}}{B(\zeta, z)} \frac{d\bar{\zeta}}{B(\zeta, z)} \right] \wedge d\zeta \\ &= \frac{1}{2} \det \left[\frac{\bar{z}}{B^*(\zeta, z)} \frac{\bar{\zeta}}{B(\zeta, z)} \frac{d\bar{\zeta}}{B(\zeta, z)} \right] \wedge d\zeta. \end{aligned} \quad (2.4)$$

Using equality (2.4) we obtain formula (2.2) for $L^\epsilon[f]$:

$$\begin{aligned} L^\epsilon[f](z) &= \frac{2}{(2\pi i)^3} \left[\int_{bU^\epsilon \times [0,1]} f(\zeta) \cdot \omega'_0 \left((1 - \lambda) \frac{\bar{z}}{B^*(\zeta, z)} + \lambda \frac{\bar{\zeta}}{B(\zeta, z)} \right) \wedge d\zeta \right. \\ & \quad \left. + \int_{U^\epsilon} f(\zeta) \cdot \omega'_0 \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \wedge d\zeta \right] \\ &= \frac{1}{(2\pi i)^3} \left[\int_{bU^\epsilon} f(\zeta) \cdot \det \left[\frac{\bar{z}}{B^*(\zeta, z)} \frac{\bar{\zeta}}{B(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta \right. \\ & \quad \left. + \int_{U^\epsilon} f(\zeta) \cdot \det \left[\frac{\bar{\zeta}}{B(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta \right]. \end{aligned}$$

□

In the following proposition we construct a Cauchy-Weil-Leray type formula on ϵ -neighborhoods of a curve in \mathbb{CP}^2 .

Proposition 2.3. *Let*

$$V = \{z \in \mathbb{CP}^2 : P(z) = 0\}$$

be a curve in \mathbb{CP}^2 , let U^ϵ be as in (1.2), and let f be a holomorphic function of homogeneity ℓ on U^ϵ . Then for an arbitrary $z \in U^\epsilon$ the following equality holds

$$f(z) = K^\epsilon[f](z), \quad (2.5)$$

where

$$\begin{aligned} K^\epsilon[f](z) = & \frac{1}{(2\pi i)^3} \left[\int_{\Gamma_2^\epsilon} f(\zeta) \cdot \det \left[\frac{\bar{z}}{B^*(\zeta, z)} \frac{\bar{\zeta}}{B(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta \right. \\ & + \int_{\Gamma_1^\epsilon} f(\zeta) \cdot \det \left[\frac{Q(\zeta, z)}{P(\zeta) - P(z)} \frac{\bar{\zeta}}{B(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta \\ & - \int_{\Gamma_{12}^\epsilon} f(\zeta) \cdot \det \left[\frac{Q(\zeta, z)}{P(\zeta) - P(z)} \frac{\bar{z}}{B^*(\zeta, z)} \frac{\bar{\zeta}}{B(\zeta, z)} \right] \wedge d\zeta \\ & \left. + \int_{U^\epsilon} f(\zeta) \cdot \det \left[\frac{\bar{\zeta}}{B(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta \right], \quad (2.6) \end{aligned}$$

$$\Gamma_1^\epsilon = \{\zeta \in \mathbb{S}^5(1) : |P(\zeta)| = \epsilon, \varrho(\zeta) < 0\}, \quad \Gamma_2^\epsilon = \{\zeta \in \mathbb{S}^5(1) : \varrho(\zeta) = 0, |P(\zeta)| < \epsilon\},$$

$$\Gamma_{12}^\epsilon = \{\zeta \in \mathbb{S}^5(1) : \varrho(\zeta) = 0, |P(\zeta)| = \epsilon\},$$

with coefficients $\{Q^i(\zeta, z)\}_{i=0}^2$ satisfying conditions (1.3) from Lemma 1.1.

The function defined by (2.6) on U^ϵ admits the descent onto a neighborhood of V in \mathbb{CP}^2 .

Proof. We begin the proof of equality (2.5) by using equality (2.2) from Lemma 2.2 with operator $K^\epsilon = L^\epsilon$:

$$\begin{aligned} K^\epsilon[f](z) = & \frac{1}{(2\pi i)^3} \left(\sum_{j=1}^2 \int_{\Gamma_j^\epsilon} f(\zeta) \cdot \det \left[\frac{\bar{z}}{B^*(\zeta, z)} \frac{\bar{\zeta}}{B(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta \right. \\ & \left. + \int_{U^\epsilon} f(\zeta) \cdot \det \left[\frac{\bar{\zeta}}{B(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta \right), \quad (2.7) \end{aligned}$$

and then transform the right-hand side of (2.7) into the right-hand side of (2.6).

To transform the integral over Γ_1^ϵ in the right-hand side of (2.7) we use equality

$$\det \begin{bmatrix} 1 & 1 & 1 & 0 \\ \eta_1(\zeta, z) & \eta_2(\zeta, z) & \frac{\bar{\zeta}}{B(\zeta, z)} & \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \end{bmatrix} = 0 \quad (2.8)$$

for $\eta_1(\zeta, z) = \frac{Q(\zeta, z)}{P(\zeta) - P(z)}$, $\eta_2(\zeta, z) = \frac{\bar{z}}{B^*(\zeta, z)}$, to obtain equality

$$\begin{aligned} & \det \left[\frac{\bar{z}}{B^*(\zeta, z)} \frac{\bar{\zeta}}{B(\zeta, z)} \frac{d\bar{\zeta}}{B(\zeta, z)} \right] \wedge d\zeta \\ & = \det \left[\frac{Q(\zeta, z)}{P(\zeta) - P(z)} \frac{\bar{\zeta}}{B(\zeta, z)} \frac{d\bar{\zeta}}{B(\zeta, z)} \right] \wedge d\zeta - \det \left[\frac{Q(\zeta, z)}{P(\zeta) - P(z)} \frac{\bar{z}}{B^*(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta, \end{aligned}$$

and, using the fact that f is holomorphic in U^ϵ , equality

$$\begin{aligned} \int_{\Gamma_1^\epsilon} f(\zeta) \cdot \det \left[\frac{\bar{z}}{B^*(\zeta, z)} \frac{\bar{\zeta}}{B(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta \\ = \int_{\Gamma_1^\epsilon} f(\zeta) \cdot \det \left[\frac{Q(\zeta, z)}{P(\zeta) - P(z)} \frac{\bar{\zeta}}{B(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta \\ - \int_{\Gamma_{12}^\epsilon} f(\zeta) \cdot \det \left[\frac{Q(\zeta, z)}{P(\zeta) - P(z)} \frac{\bar{z}}{B^*(\zeta, z)} \frac{\bar{\zeta}}{B(\zeta, z)} \right] \wedge d\zeta. \end{aligned} \quad (2.9)$$

Then using equality (2.9) in (2.7) we obtain equality (2.6). \square

3. RESIDUAL INTEGRALS ON V .

In this section we analyze the behavior of the integrals in the right-hand side of (2.6) as $\epsilon \rightarrow 0$. In all estimates below we use the same notation “ C ” for different constants.

For the limit of the first integral in the right-hand side of (2.6) we have the following lemma.

Lemma 3.1. *Let $z \in U^{\epsilon_0} \subset \mathbb{S}^5(1)$ for some $\epsilon_0 > 0$, and let f be a holomorphic function in U^{ϵ_0} . Then*

$$\lim_{\epsilon \rightarrow 0} \int_{\Gamma_2^\epsilon} f(\zeta) \cdot \det \left[\frac{\bar{z}}{B^*(\zeta, z)} \frac{\bar{\zeta}}{B(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta = 0. \quad (3.1)$$

Proof. Using the uniform boundedness of the kernel on Γ_2^ϵ for $z \in V$ we obtain the following estimate

$$\left| \int_{\Gamma_2^\epsilon} f(\zeta) \cdot \det \left[\frac{\bar{z}}{B^*(\zeta, z)} \frac{\bar{\zeta}}{B(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta \right| \leq C \cdot \text{Volume} \{ \Gamma_2^\epsilon \} \leq C \cdot \epsilon \rightarrow 0$$

as $\epsilon \rightarrow 0$. \square

For the fourth integral in the right-hand side of (2.6) we have the following lemma.

Lemma 3.2. *Let $z \in U^{\epsilon_0} \subset \mathbb{S}^5(1)$ for some $\epsilon_0 > 0$, and let f be a holomorphic function in U^{ϵ_0} . Then*

$$\lim_{\epsilon \rightarrow 0} \int_{U^\epsilon} f(\zeta) \cdot \det \left[\frac{\bar{\zeta}}{B(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta = 0. \quad (3.2)$$

Proof. To prove equality (3.2) we represent the integral in the left-hand side as

$$\begin{aligned} \int_{U^\epsilon} f(\zeta) \cdot \det \left[\frac{\bar{\zeta}}{B(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta \\ = \int_{U^\epsilon \cap \{ |B(\zeta, z)| > \epsilon \}} f(\zeta) \cdot \det \left[\frac{\bar{\zeta}}{B(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta \\ + \int_{U^\epsilon \cap \{ |B(\zeta, z)| < \epsilon \}} (f(\zeta) - f(z)) \cdot \det \left[\frac{\bar{\zeta}}{B(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta \\ + f(z) \int_{U^\epsilon \cap \{ |B(\zeta, z)| < \epsilon \}} \det \left[\frac{\bar{\zeta}}{B(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta. \end{aligned} \quad (3.3)$$

Then we obtain the following estimate for the first integral in the right-hand side of (3.3)

$$\begin{aligned} & \left| \int_{U^\epsilon \cap \{|B(\zeta, z)| > \epsilon\}} f(\zeta) \cdot \det \left[\frac{\bar{\zeta}}{B(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta \right| \\ & \leq C \cdot \int_0^A dt \int_0^\epsilon du \int_0^\epsilon dv \int_0^A \frac{r dr}{(\epsilon + t + u^2 + v^2 + r^2)^3} \leq C \cdot \int_0^\epsilon \frac{\tau d\tau}{\epsilon + \tau^2} \\ & \leq C \cdot \int_0^{\epsilon^2} \frac{ds}{\epsilon + s} \leq C \cdot \epsilon, \quad (3.4) \end{aligned}$$

where we denoted $\tau = \sqrt{u^2 + v^2}$, $s = \tau^2$.

For the second integral in the right-hand side of (3.3) we have

$$\begin{aligned} & \left| \int_{U^\epsilon \cap \{|B(\zeta, z)| < \epsilon\}} (f(\zeta) - f(z)) \cdot \det \left[\frac{\bar{\zeta}}{B(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta \right| \\ & \leq C \cdot \int_0^A dt \int_0^A du \int_0^A dv \int_0^{\sqrt{\epsilon}} \frac{r^2 dr}{(t + u^2 + v^2 + r^2)^3} \leq C \cdot \int_0^{\sqrt{\epsilon}} dr \leq C \cdot \sqrt{\epsilon}. \quad (3.5) \end{aligned}$$

Using estimates (3.4), (3.5) and equality

$$\lim_{\delta \rightarrow 0} \left| \int_{|B(\zeta, z)| < \delta} \det \left[\frac{\bar{\zeta}}{B(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta \right| = 0$$

we obtain equality (3.2). \square

Using now the barrier function (1.4) we estimate the behavior of the third integral in the right-hand side of (2.6) in the following proposition.

Proposition 3.3. *Let f be a holomorphic function of homogeneity $\ell < 0$ on U^ϵ for some $\epsilon > 0$. Then for $z \in V$ the following equality holds*

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \int_{\Gamma_{12}^\epsilon} f(\zeta) \cdot \det \left[\frac{Q(\zeta, z)}{P(\zeta) - P(z)} \frac{\bar{z}}{B^*(\zeta, z)} \frac{\bar{\zeta}}{B(\zeta, z)} \right] d\zeta \\ & = - \lim_{\epsilon \rightarrow 0} \int_{\Gamma_{12}^\epsilon} f(\zeta) \cdot \det \left[\frac{R(z)}{F(z, \zeta)} \frac{Q(\zeta, z)}{P(\zeta) - P(z)} \frac{\bar{\zeta}}{B(\zeta, z)} \right] d\zeta, \quad (3.6) \end{aligned}$$

where $F(z, \zeta) = \sum_{i=0}^2 R_i(z)(\zeta_i - z_i)$ is the barrier function from (1.4).

Proof. Using equality

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ \frac{Q(\zeta, z)}{P(\zeta) - P(z)} & \frac{\bar{z}}{B^*(\zeta, z)} & \frac{\bar{\zeta}}{B(\zeta, z)} & \frac{R(z)}{F(z, \zeta)} \end{bmatrix} = 0,$$

its corollary

$$\begin{aligned} & \det \left[\frac{\bar{z}}{B^*(\zeta, z)} \frac{\bar{\zeta}}{B(\zeta, z)} \frac{R(z)}{F(z, \zeta)} \right] - \det \left[\frac{Q(\zeta, z)}{P(\zeta) - P(z)} \frac{\bar{\zeta}}{B(\zeta, z)} \frac{R(z)}{F(z, \zeta)} \right] \\ & + \det \left[\frac{Q(\zeta, z)}{P(\zeta) - P(z)} \frac{\bar{z}}{B^*(\zeta, z)} \frac{R(z)}{F(z, \zeta)} \right] - \det \left[\frac{Q(\zeta, z)}{P(\zeta) - P(z)} \frac{\bar{z}}{B^*(\zeta, z)} \frac{\bar{\zeta}}{B(\zeta, z)} \right] = 0, \end{aligned}$$

and estimate

$$\left| \int_{\Gamma_{12}^\epsilon} f(\zeta) \cdot \det \left[\frac{\bar{z}}{B^*(\zeta, z)} \frac{\bar{\zeta}}{B(\zeta, z)} \frac{R(z)}{F(z, \zeta)} \right] \right| \leq C \cdot \epsilon \rightarrow 0$$

for $z \in V$ as $\epsilon \rightarrow 0$, we obtain equality

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{\Gamma_{12}^\epsilon} f(\zeta) \cdot \det \left[\frac{Q(\zeta, z)}{P(\zeta) - P(z)} \frac{\bar{z}}{B^*(\zeta, z)} \frac{\bar{\zeta}}{B(\zeta, z)} \right] d\zeta \\ = - \lim_{\epsilon \rightarrow 0} \int_{\Gamma_{12}^\epsilon} f(\zeta) \cdot \det \left[\frac{Q(\zeta, z)}{P(\zeta) - P(z)} \frac{\bar{\zeta}}{B(\zeta, z)} \frac{R(z)}{F(z, \zeta)} \right] d\zeta \\ + \lim_{\epsilon \rightarrow 0} \int_{\Gamma_{12}^\epsilon} f(\zeta) \cdot \det \left[\frac{Q(\zeta, z)}{P(\zeta) - P(z)} \frac{\bar{z}}{B^*(\zeta, z)} \frac{R(z)}{F(z, \zeta)} \right] d\zeta. \end{aligned} \quad (3.7)$$

To simplify the right-hand side of equality (3.7) we need the following lemma.

Lemma 3.4. *If f is a holomorphic function of homogeneity $\ell < 0$ on U^ϵ for some $\epsilon > 0$, then for $z \in V$*

$$\int_{\Gamma_{12}^\epsilon} f(\zeta) \cdot \det \left[\frac{Q(\zeta, z)}{P(\zeta) - P(z)} \frac{\bar{z}}{B^*(\zeta, z)} \frac{R(z)}{F(z, \zeta)} \right] d\zeta = 0. \quad (3.8)$$

Proof. To evaluate for $z \in V$ the integral

$$\begin{aligned} \int_{\Gamma_{12}^\epsilon} f(\zeta) \cdot \det \left[\frac{Q(\zeta, z)}{P(\zeta) - P(z)} \frac{\bar{z}}{B^*(\zeta, z)} \frac{R(z)}{F(z, \zeta)} \right] d\zeta \\ = \int_{\Gamma_{12}^\epsilon} f(\zeta) \cdot \det \left[\frac{Q(\zeta, z)}{P(\zeta)} \frac{\bar{z}}{B^*(\zeta, z)} \frac{R(z)}{F(z, \zeta)} \right] d\zeta \end{aligned}$$

we consider the restriction of the function f to V , and its lift to $\sigma^{-1}(V) \subset \sigma^{-1}(\mathbb{CP}^2) = \mathbb{C}^3 \setminus \{0\}$, which we also denote by f .

We use the following estimates for $|\zeta| \rightarrow \infty$

$$\left| \frac{\bar{z}}{B^*(\zeta, z)} \right|, \left| \frac{Q(\zeta, z)}{P(\zeta)} \right|, \left| \frac{R(z)}{F(z, \zeta)} \right| \leq \frac{C}{|\zeta|}, \quad (3.9)$$

and notice that for all values of $a > 0$ the sets

$$\Gamma_{12}^\epsilon(a) = \left\{ \zeta \in \mathbb{S}^5(a) : |P(\zeta)| = \epsilon \cdot a^{\deg P}, \varrho(\zeta) = 0 \right\}$$

are real analytic subvarieties of $\mathbb{S}^5(a)$ of real dimension 3 satisfying

$$c \cdot a^3 \cdot \text{Volume}(\Gamma_{12}^\epsilon) < \text{Volume}(\Gamma_{12}^\epsilon(a)) < C \cdot a^3 \cdot \text{Volume}(\Gamma_{12}^\epsilon). \quad (3.10)$$

We use the Stokes' formula for the form

$$f(\zeta) \cdot \det \left[\frac{\bar{z}}{B^*(\zeta, z)} \frac{Q(\zeta, z)}{P(\zeta)} \frac{R(z)}{F(z, \zeta)} \right] d\zeta,$$

which is holomorphic with respect to ζ , on the variety

$$\left\{ \zeta \in \mathbb{C}^3 : |P(\zeta)| = \epsilon \cdot |\zeta|^{\deg P}, \varrho(\zeta) = 0, 1 < |\zeta| < a \right\}$$

with the boundary

$$\Gamma_{12}^\epsilon \cup \Gamma_{12}^\epsilon(a).$$

Then for f satisfying $f(\lambda\zeta) = \lambda^\ell f(\zeta)$ with $\ell < 0$ and $z \in V$, using estimates (3.9) and (3.10) we obtain the following estimate

$$\begin{aligned} & \left| \int_{\Gamma_{12}^\epsilon} f(\zeta) \cdot \det \left[\frac{Q(\zeta, z)}{P(\zeta)} \frac{\bar{z}}{B^*(\zeta, z)} \frac{R(z)}{F(z, \zeta)} \right] d\zeta \right| \\ &= \left| \int_{\Gamma_{12}^\epsilon(a)} f(\zeta) \cdot \det \left[\frac{Q(\zeta, z)}{P(\zeta)} \frac{\bar{z}}{B^*(\zeta, z)} \frac{R(z)}{F(z, \zeta)} \right] d\zeta \right| \leq C \cdot \frac{\text{Volume}(\Gamma_{12}^\epsilon(a))}{a^4} \rightarrow 0 \end{aligned}$$

as $a \rightarrow \infty$. \square

Applying Lemma 3.4 to the second integral in the right-hand side of equality (3.7), we obtain equality (3.6). \square

To estimate the second integral of the right-hand side of (2.6) we use the following proposition.

Proposition 3.5. *For a holomorphic function f on U^ϵ and $z \in V$ the following equality holds*

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \int_{\Gamma_1^\epsilon} f(\zeta) \cdot \det \left[\frac{Q(\zeta, z)}{P(\zeta) - P(z)} \frac{\bar{\zeta}}{B(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta \\ &= - \lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \int_{\Gamma_{12}^{\epsilon, \delta}(z)} f(\zeta) \cdot \det \left[\frac{R(z)}{F(z, \zeta)} \frac{Q(\zeta, z)}{P(\zeta) - P(z)} \frac{\bar{\zeta}}{B(\zeta, z)} \right] d\zeta \\ &\quad - \lim_{\epsilon \rightarrow 0} \int_{\Gamma_{12}^\epsilon} f(\zeta) \cdot \det \left[\frac{R(z)}{F(z, \zeta)} \frac{Q(\zeta, z)}{P(\zeta) - P(z)} \frac{\bar{\zeta}}{B(\zeta, z)} \right] d\zeta \quad (3.11) \end{aligned}$$

where

$$\Gamma_{12}^{\epsilon, \delta}(z) = \{ \zeta \in \mathbb{S}^5(1) : |P(\zeta)| = \epsilon, |F(z, \zeta)| = \delta \}.$$

Proof. Using equality (2.8) with $\eta_1(\zeta, z) = \frac{R(z)}{F(z, \zeta)}$, $\eta_2(\zeta, z) = \frac{Q(\zeta, z)}{P(\zeta) - P(z)}$, we obtain equality

$$\begin{aligned} & \det \left[\frac{Q(\zeta, z)}{P(\zeta) - P(z)} \frac{\bar{\zeta}}{B(\zeta, z)} \frac{d\bar{\zeta}}{B(\zeta, z)} \right] \wedge d\zeta \\ &= \det \left[\frac{R(z)}{F(z, \zeta)} \frac{\bar{\zeta}}{B(\zeta, z)} \frac{d\bar{\zeta}}{B(\zeta, z)} \right] \wedge d\zeta - \det \left[\frac{R(z)}{F(z, \zeta)} \frac{Q(\zeta, z)}{P(\zeta) - P(z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta \\ &= \det \left[\frac{R(z)}{F(z, \zeta)} \frac{\bar{\zeta}}{B(\zeta, z)} \frac{d\bar{\zeta}}{B(\zeta, z)} \right] \wedge d\zeta - \bar{\partial} \left(\det \left[\frac{R(z)}{F(z, \zeta)} \frac{Q(\zeta, z)}{P(\zeta) - P(z)} \frac{\bar{\zeta}}{B(\zeta, z)} \right] \right) \wedge d\zeta. \end{aligned}$$

Then, using the Stokes' formula on

$$\Gamma_1^{\epsilon, \delta}(z) = \{ \zeta \in \mathbb{S}^5(1) : |P(\zeta)| = \epsilon, |F(z, \zeta)| > \delta \}$$

for the second term of the right-hand side of equality above, and holomorphic dependence of $f(\zeta)$ on ζ we obtain the following equality

$$\begin{aligned}
& \int_{\Gamma_1^\epsilon} f(\zeta) \cdot \det \left[\frac{Q(\zeta, z)}{P(\zeta) - P(z)} \frac{\bar{\zeta}}{B(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta \\
&= \lim_{\delta \rightarrow 0} \int_{\Gamma_1^{\epsilon, \delta}(z)} f(\zeta) \cdot \det \left[\frac{Q(\zeta, z)}{P(\zeta) - P(z)} \frac{\bar{\zeta}}{B(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta \\
&= \lim_{\delta \rightarrow 0} \int_{\Gamma_1^{\epsilon, \delta}(z)} f(\zeta) \cdot \det \left[\frac{R(z)}{F(z, \zeta)} \frac{\bar{\zeta}}{B(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta \\
&\quad - \lim_{\delta \rightarrow 0} \int_{\Gamma_{12}^{\epsilon, \delta}(z)} f(\zeta) \cdot \det \left[\frac{R(z)}{F(z, \zeta)} \frac{Q(\zeta, z)}{P(\zeta) - P(z)} \frac{\bar{\zeta}}{B(\zeta, z)} \right] d\zeta \\
&\quad - \int_{\Gamma_{12}^\epsilon} f(\zeta) \cdot \det \left[\frac{R(z)}{F(z, \zeta)} \frac{Q(\zeta, z)}{P(\zeta) - P(z)} \frac{\bar{\zeta}}{B(\zeta, z)} \right] d\zeta. \quad (3.12)
\end{aligned}$$

In the following lemma we estimate the behavior of the first integral in the right-hand side of (3.12) as $\epsilon \rightarrow 0$.

Lemma 3.6. *Let f be a holomorphic function on U^ϵ for some $\epsilon > 0$. Then for $z \in V$ we have*

$$\lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \int_{\Gamma_1^{\epsilon, \delta}} f(\zeta) \cdot \det \left[\frac{R(z)}{F(z, \zeta)} \frac{\bar{\zeta}}{B(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta = 0. \quad (3.13)$$

Proof. Denoting

$$U_2^{\epsilon, \delta}(z) = \{ \zeta \in \mathbb{S}^5(1) : |F(z, \zeta)| > \delta, |P(\zeta)| < \epsilon \},$$

$$\Gamma_2^{\epsilon, \delta}(z) = \{ \zeta \in \mathbb{S}^5(1) : |F(z, \zeta)| = \delta, |P(\zeta)| < \epsilon \},$$

we apply the Stokes' formula to the form

$$f(\zeta) \cdot \det \left[\frac{R(z)}{F(z, \zeta)} \frac{\bar{\zeta}}{B(\zeta, z)} \frac{d\bar{\zeta}}{B(\zeta, z)} \right] \wedge d\zeta$$

on the domain $U_2^{\epsilon, \delta}(z)$. Then we obtain equality

$$\begin{aligned}
& \int_{\Gamma_1^{\epsilon, \delta}} f(\zeta) \cdot \det \left[\frac{R(z)}{F(z, \zeta)} \frac{\bar{\zeta}}{B(\zeta, z)} \frac{d\bar{\zeta}}{B(\zeta, z)} \right] \wedge d\zeta \\
&+ \int_{\Gamma_2^\epsilon} f(\zeta) \cdot \det \left[\frac{R(z)}{F(z, \zeta)} \frac{\bar{\zeta}}{B(\zeta, z)} \frac{d\bar{\zeta}}{B(\zeta, z)} \right] \wedge d\zeta \\
&+ \int_{\Gamma_2^{\epsilon, \delta}(z)} f(\zeta) \cdot \det \left[\frac{R(z)}{F(z, \zeta)} \frac{\bar{\zeta}}{B(\zeta, z)} \frac{d\bar{\zeta}}{B(\zeta, z)} \right] \wedge d\zeta \\
&= \int_{U_2^{\epsilon, \delta}(z)} f(\zeta) \cdot \det \left[\frac{R(z)}{F(z, \zeta)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta. \quad (3.14)
\end{aligned}$$

For the third integral in the left-hand side of (3.14) we have

$$\begin{aligned}
& \left| \int_{\Gamma_2^{\epsilon, \delta}(z)} f(\zeta) \cdot \det \left[\frac{R(z)}{F(z, \zeta)} \frac{\bar{\zeta}}{B(\zeta, z)} \frac{d\bar{\zeta}}{B(\zeta, z)} \right] \wedge d\zeta \right| \leq C \cdot \left| \int_0^A dt \int_0^\epsilon d\rho \int_{|w|=\delta} \frac{\rho dw}{\delta \cdot (\epsilon + t + \rho^2)^2} \right| \\
&\leq C \cdot \int_0^\epsilon \frac{\rho d\rho}{(\epsilon + \rho^2)} \leq C \cdot \int_0^{\epsilon^2} \frac{du}{\epsilon + u} \rightarrow 0
\end{aligned}$$

as $\epsilon \rightarrow 0$.

For the second integral in the left-hand side of (3.14) using uniform boundedness of the kernel on Γ_2^ϵ for $z \in V$ we obtain the following estimate

$$\left| \int_{\Gamma_2^\epsilon} f(\zeta) \cdot \det \left[\frac{R(z)}{F(z, \zeta)} \frac{\bar{\zeta}}{B(\zeta, z)} \frac{d\bar{\zeta}}{B(\zeta, z)} \right] \wedge d\zeta \right| \leq C \cdot \text{Volume} \{ \Gamma_2^\epsilon \} \leq C \cdot \epsilon \rightarrow 0$$

as $\epsilon \rightarrow 0$.

For the integral in the right-hand side of equality (3.14) we use the equality

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \left| \int_{U_{2^{\epsilon, \delta}}(z)} f(\zeta) \cdot \det \left[\frac{R(z)}{F(z, \zeta)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta \right| \\ = \lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \left| \int_{U_{2^{\epsilon, \delta}}(z)} f(\zeta) \cdot \det \left[\frac{\bar{\zeta}}{B(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta \right| = 0, \end{aligned}$$

where in the second equality we used Lemma 3.2.

From the above estimates follows equality (3.13). \square

Using equality (3.13) in (3.12) we obtain equality (3.11) from Proposition 3.5. \square

4. PROOF OF THEOREM 1.

Combining Propositions 3.3 and 3.5 we obtain from formula (2.6) the following formula for $z \in V$:

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} K^\epsilon[f](z) &= \frac{1}{(2\pi i)^3} \lim_{\epsilon \rightarrow 0} \left[\int_{\Gamma_1^\epsilon} f(\zeta) \cdot \det \left[\frac{Q(\zeta, z)}{P(\zeta)} \frac{\bar{\zeta}}{B(\zeta, z)} \bar{\partial} \left(\frac{\bar{\zeta}}{B(\zeta, z)} \right) \right] \wedge d\zeta \right. \\ &\quad \left. - \int_{\Gamma_{12}^\epsilon} f(\zeta) \cdot \det \left[\frac{Q(\zeta, z)}{P(\zeta)} \frac{\bar{z}}{B^*(\zeta, z)} \frac{\bar{\zeta}}{B(\zeta, z)} \right] d\zeta \right] \\ &= \frac{1}{(2\pi i)^3} \lim_{\epsilon \rightarrow 0} \left[- \lim_{\delta \rightarrow 0} \int_{\Gamma_{12}^{\epsilon, \delta}(z)} f(\zeta) \cdot \det \left[\frac{R(z)}{F(z, \zeta)} \frac{Q(\zeta, z)}{P(\zeta)} \frac{\bar{\zeta}}{B(\zeta, z)} \right] d\zeta \right. \\ &\quad \left. + 2 \int_{\Gamma_{12}^\epsilon} f(\zeta) \cdot \det \left[\frac{Q(\zeta, z)}{P(\zeta)} \frac{R(z)}{F(z, \zeta)} \frac{\bar{\zeta}}{B(\zeta, z)} \right] d\zeta \right]. \quad (4.1) \end{aligned}$$

We further transform formula (4.1) into an integral formula with integral taken over the boundary and the kernel holomorphically depending on z by explicitly computing limits in the first integral of its right-hand side in the following lemma.

Lemma 4.1. *Let $z \in V$ be fixed, let $w \in \mathcal{V}_z$, and let f be a holomorphic function on U^ϵ . Then the following equality holds*

$$\frac{1}{(2\pi i)^3} \lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \int_{\Gamma_{12}^{\epsilon, \delta}(w)} f(\zeta) \cdot \det \left[\frac{R(w)}{F(w, \zeta)} \frac{Q(\zeta, w)}{P(\zeta)} \frac{\bar{\zeta}}{B(\zeta, w)} \right] d\zeta = \sum_{j=0}^p f(w^{(j)}), \quad (4.2)$$

where $w_0 = w, w_1, \dots, w_p$ are the points from (1.5) (ii) satisfying equality $F(w, w^{(j)}) = 0$.

Proof. We represent the integral in the left-hand side of equality (4.2) as

$$\begin{aligned} & \frac{1}{(2\pi i)^3} \lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \int_{\Gamma_{12}^{\epsilon, \delta}(w)} f(\zeta) \cdot \det \left[\frac{R(w)}{F(w, \zeta)} \frac{Q(\zeta, w)}{P(\zeta)} \frac{\bar{\zeta}}{B(\zeta, w)} \right] d\zeta \\ &= \sum_{j=0}^p \frac{1}{(2\pi i)^3} \lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \int_{\Gamma_{12}^{\epsilon, \delta}(w^{(j)})} f(\zeta) \cdot \det \left[\frac{R(w)}{F(w, \zeta)} \frac{Q(\zeta, w)}{P(\zeta)} \frac{\bar{\zeta}}{B(\zeta, w)} \right] d\zeta, \end{aligned} \quad (4.3)$$

where $\Gamma_{12}^{\epsilon, \delta}(w^{(j)}) = \Gamma_{12}^{\epsilon, \delta}(w) \cap U_j$, and U_j is a small enough neighborhood of the point $w^{(j)}$.

For each $w^{(j)}$ ($j = 0, \dots, p$) we have

$$\begin{aligned} & \frac{1}{(2\pi i)^3} \lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \int_{\Gamma_{12}^{\epsilon, \delta}(w^{(j)})} f(\zeta) \cdot \det \left[\frac{R(w)}{F(w, \zeta)} \frac{Q(\zeta, w)}{P(\zeta)} \frac{\bar{\zeta}}{B(\zeta, w)} \right] d\zeta \\ &= \frac{1}{(2\pi i)^3} \lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \int_{\Gamma_{12}^{\epsilon, \delta}(w^{(j)})} f(\zeta) \frac{d_\zeta F(w^{(j)}, \zeta)}{F(w^{(j)}, \zeta)} \wedge \frac{d_\zeta P(\zeta)}{P(\zeta)} \wedge \frac{d_\zeta B(\zeta, w^{(j)})}{B(\zeta, w^{(j)})} \\ &= \frac{1}{(2\pi i)} \int_{\mathbb{C}^1(w^{(j)}) \cap \{|\zeta|=1\}} f(\zeta) \cdot \frac{d_\zeta B(\zeta, w^{(j)})}{B(\zeta, w^{(j)})}, \end{aligned} \quad (4.4)$$

where in the last integral we integrate over the unit circle in

$$\mathbb{C}^1(w^{(j)}) = \left\{ \zeta = \mu \cdot w^{(j)} \right\}.$$

To evaluate the last integral we substitute variables $\zeta_1 = \lambda \zeta_0$, $\zeta_2 = \lambda \zeta_0$ and obtain equality

$$B(\zeta, z) = \bar{\zeta}_0(\zeta_0 - w_0) + \bar{\lambda}_1 \lambda_1 \bar{\zeta}_0(\zeta_0 - w_0) + \bar{\lambda}_2 \lambda_2 \bar{\zeta}_0(\zeta_0 - w_0) = \bar{\zeta}_0(\zeta_0 - w_0)(1 + \bar{\lambda}_1 \lambda_1 + \bar{\lambda}_2 \lambda_2).$$

Using this equality in equality (4.4) we obtain

$$\frac{1}{(2\pi i)} \int_{\mathbb{C}^1(w^{(j)}) \cap \{|\zeta|=1\}} f(\zeta) \cdot \frac{d_\zeta B(\zeta, w^{(j)})}{B(\zeta, w^{(j)})} = \frac{1}{(2\pi i)} \int_{\mathbb{C}^1(w^{(j)}) \cap \{|\zeta|=1\}} f(\zeta) \cdot \frac{d\zeta_0}{\zeta_0 - w_0^{(j)}} = f(w^{(j)}).$$

□

Using equality (4.2) in formula (4.1) we obtain for $w \in \mathcal{V}_z$ equality

$$\lim_{\epsilon \rightarrow 0} K^\epsilon[f](w) = - \sum_{j=0}^p f(w^{(j)}) + \frac{2}{(2\pi i)^3} \lim_{\epsilon \rightarrow 0} \int_{\Gamma_{12}^\epsilon} f(\zeta) \cdot \det \left[\frac{Q(\zeta, w)}{P(\zeta)} \frac{R(w)}{F(w, \zeta)} \frac{\bar{\zeta}}{B(\zeta, w)} \right] d\zeta,$$

and combining it with (2.5) obtain equality

$$2f(w) + \sum_{j=1}^p f(w^{(j)}) = \frac{2}{(2\pi i)^3} \lim_{\epsilon \rightarrow 0} \int_{\Gamma_{12}^\epsilon} f(\zeta) \cdot \det \left[\frac{Q(\zeta, w)}{P(\zeta)} \frac{R(w)}{F(w, \zeta)} \frac{\bar{\zeta}}{B(\zeta, w)} \right] d\zeta. \quad (4.5)$$

Using equality (4.5) we construct the following system of linear equations with Vandermonde matrix for the values of the function f at the points $\{w, w^{(1)}, \dots, w^{(p)}\}$:

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ \frac{w_1}{w_0} & \frac{w_1^{(1)}}{w_0} & \dots & \frac{w_1^{(p)}}{w_0^{(p)}} \\ \vdots & \vdots & & \vdots \\ \left(\frac{w_1}{w_0}\right)^p & \left(\frac{w_1^{(1)}}{w_0}\right)^p & \dots & \left(\frac{w_1^{(p)}}{w_0^{(p)}}\right)^p \end{bmatrix} \begin{bmatrix} 2f(w) \\ f(w^{(1)}) \\ \vdots \\ f(w^{(p)}) \end{bmatrix} = \begin{bmatrix} G_0(w) \\ G_1(w) \\ \vdots \\ G_p(w) \end{bmatrix}, \quad (4.6)$$

where

$$G_k(w) = \frac{2}{(2\pi i)^3} \lim_{\epsilon \rightarrow 0} \int_{\Gamma_{12}^\epsilon} f(\zeta) \cdot \left(\frac{\zeta_1}{\zeta_0} \right)^k \cdot \det \left[\frac{Q(\zeta, w)}{P(\zeta)} \frac{R(w)}{F(w, \zeta)} \frac{\bar{\zeta}}{B(\zeta, w)} \right] d\zeta.$$

By summing individual systems of the form (4.6)

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \frac{w_1}{w_0} & \frac{w_1^{(1)}}{w_0} & \cdots & \frac{w_1^{(p)}}{w_0^{(p)}} \\ \vdots & \vdots & & \vdots \\ \left(\frac{w_1}{w_0} \right)^p & \left(\frac{w_1^{(1)}}{w_0} \right)^p & \cdots & \left(\frac{w_1^{(p)}}{w_0^{(p)}} \right)^p \end{bmatrix} \begin{bmatrix} f(w) \\ \vdots \\ 2f(w^{(j)}) \\ \vdots \\ f(w^{(p)}) \end{bmatrix} = \begin{bmatrix} G_0(w^{(j)}) \\ \vdots \\ G_j(w^{(j)}) \\ \vdots \\ G_p(w^{(j)}) \end{bmatrix}, \quad (4.7)$$

we construct a “symmetrized” version of (4.6)

$$A(w) \cdot f(w) = \frac{p+1}{p+2} \cdot G(w, w^{(1)}, \dots, w^{(p)}), \quad (4.8)$$

where A is the Vandermonde matrix of system (4.6), and

$$G_k(w, w^{(1)}, \dots, w^{(p)}) = \sum_{j=0}^p G_k(w^{(j)}). \quad (4.9)$$

Then using the Cramer’s rule for system (4.8) we obtain equality (1.6).

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